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Best Approximation in Certain Classes of Normed Linear Spaces

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INTRODUCTION

Some results on best approximation in concrete spaces, e.g., the space of continuous functions or the space of integrable functions, lead us to introduce in [8, 10] two classes of normed linear spaces called "with property (C)" [8] and "with property (A)" [10], which seems to be particularly well suited for applications to best approximation. To support this idea, we present here much new material which, we hope, will convince the reader of the usefulness of these classes. From our general results proved for a space which belongs to one or both of these classes, we derive many known results for the concrete spaces which are contained in [1, 5, 6, 8, 12, 14, 15, 17, 19, 21]. We notice here that these results for the concrete spaces referred to above are sometimes formulated and always proved, using the specific properties of the spaces under consideration.

Another way of generalization concerns the approximant set. In this paper we consider best approximation by elements of suns. As is well known, any convex set is a sun, but the converse is not true. Hence some results on best approximation by elements of linear subspaces or convex sets in the concrete spaces can be extended to suns, using our results.

On the other hand, the simple fact that a concrete space belongs to one or both of these classes furnishes (by the definitions of these classes) geometrical properties of that space. In particular, we obtain common geometrical properties for the main concrete spaces $C(Q)$, $C_0(T)$, $L^1(T,\mu)$, $L^\infty(T,\mu)$.

This paper contains four sections, Section 1 being a presentation of known results or facts from functional analysis and the theory of best approximation, as well as the terminology and notations necessary for an easy understanding of the other sections. Section 2, resp. Section 3, deals with the class of spaces with property (C), resp. (A). The main result of Section 2

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gives a necessary and sufficient condition in order that a sun in a space with property (C) be semi-Chebyshev. This result is used in Section 4, which deals with the class of spaces having both property (C) and property (A), to prove a characterization of a semi-Chebyshev sun by the strict Kolmogorov criterion. Applications are given throughout these last three sections.

In this paper we consider real normed linear spaces, but the results can be extended in the usual way to the complex normed linear spaces.

1. Preliminaries

Throughout this paper E will stand for a *real* normed linear space, its unit sphere being denoted by S_E, and its conjugate space by E^*. For $x, y \in E$ let us denote by $\tau(x, y)$ the one-sided Gateaux differential at x in the direction y, i.e.,

$$\tau(x, y) = \lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}.$$

It is well known (see, e.g., [7]) that

$$\tau(x, y) + \tau(x, -y) \geq 0. \tag{1.1}$$

For each $x \in E$ we denote

$$N(x) = N_E(x) = \{y \in E \mid \tau(x, y) + \tau(x, -y) = 0\},$$

$$A(x) = A_E(x) = \{f \in E^* \mid f(x) = \|x\|, \|f\| = 1\}.$$

Then by [11, 8] we have

$$N(x) = A(x)_{\perp} \dotplus [x] \tag{1.2}$$

and by [16] we have

$$A(x)_{\perp} = \{y \in E \mid \tau(x, y) = \tau(x, -y) = 0\}, \tag{1.3}$$

where for a set $A \subset E^*$, $A_{\perp} = \{x \in E \mid f(x) = 0$ for each $f \in A\}$.

Summarizing some known facts in the concrete spaces (see [11, 8, 10] for formulas (1.4), (1.5) and [10] for (1.6)–(1.8)), we have the following useful formulas.

Let $C(Q)$ be the Banach space of all real-valued continuous functions over the compact Hausdorff space Q with the sup norm, and for $x \in C(Q)$ let us denote by $\operatorname{crit} x = \{q \in Q \mid |x(q)| = \|x\|\}$ and $Z(x) = \{q \in Q \mid x(q) = 0\}$. Then

$$N(x) = \{y \in C(Q) \mid \operatorname{crit} x \subset Z(y)\} \dotplus [x]. \tag{1.4}$$

When (T, μ) is a positive measure space, let $L^1(T, \mu)$ be the Banach space of the equivalence classes of measurable real-valued functions x on T for which $\|x\| = \int_T |x| \, d\mu < \infty$, and for $x \in L^1(T, \mu)$ we denote by $Z(x) = \{t \in T \mid x(t) = 0\}$ (defined up to a μ-null set). Then

$$N(x) = \{y \in L^1(T, \mu) \mid Z(x) \subset Z(y) \text{ a.e.}\} \tag{1.5}$$

and for each $x, y \in L^1(T, \mu)$ we have

$$\frac{\tau(x, y) - \tau(x, -y)}{2} = \int_{T \setminus Z(x)} y \operatorname{sign} x \, d\mu. \tag{1.6}$$

$$\operatorname{dist}(y, N(x)) = \int_{Z(x)} |y| \, d\mu. \tag{1.7}$$

When Q is a compact Hausdorff space and ν a positive Radon measure on Q such that the support of ν is Q, let $C^1(Q, \nu)$ be the linear subspace of $L^1(Q, \nu)$, of the equivalence classes of real-valued continuous functions on Q, with the norm $\|x\| = \int_Q |x| \, d\nu$. Then for each $x \in C^1(Q, \nu)$ we have [10] that $N_{C^1(Q,\nu)}(x)$ is dense in $N_{L^1(Q,\nu)}(x)$ and (1.5)–(1.7) are true, replacing L^1 by C^1, T by Q and μ by ν.

Let l^∞ be the Banach space of all real bounded sequences, endowed with the sup norm, and for $x = (\xi_n) \in l^\infty$, let $I_x = \{n \mid |\xi_n| = \|x\|\}$ and let $\mathcal{J}_x$ be the set of all sequences (n_k) such that $\lim \xi_{n_k} = \pm\|x\|$, $-\|x\| < \xi_{n_k} < \|x\|$. Then

$$N(x) = \{y = (\eta_n) \in l^\infty \mid \eta_n = 0, n \in I_x, \lim \eta_{n_k} = 0, (n_k) \in \mathcal{J}_x\} \oplus |x|. \tag{1.8}$$

The last part of this section contains the background material from best approximation theory, which will be used in the other sections.

Let G be a nonempty subset of E, and for $x \in E$ let us denote by $P_G(x)$ the set of all best approximations of x out of G, i.e.,

$$P_G(x) = \{g_0 \in G \mid \|x - g_0\| = \operatorname{dist}(x, G)\}.$$

The set G is called: (1) *proximinal* if $P_G(x) \neq \varnothing$ for each $x \in E$; (2) *semi-Chebyshev* if $P_G(x)$ contains at most one element for each $x \in E$; (3) *Chebyshev* if $P_G(x)$ contains exactly one element for each $x \in E$.

An element $g_0 \in G$ is called a *strongly unique* element of best approximation of $x \in E$ if there exists a number $\rho > 0$ such that for each $g \in G$, $\|x - g\| \geqslant \|x - g_0\| + \rho \|g - g_0\|$. G is said to be a *strongly Chebyshev* set, if each $x \in E$ has a strongly unique element of best approximation in G. Clearly, if $g_0 \in G$ is a strongly unique element of best approximation of x in G, then $P_G(x) = \{g_0\}$. The converse is not always true.

For $x \in E\setminus\bar{G}$ and $g_0 \in G$, we recall that the pair (g_0, x) is said to satisfy the *Kolmogorov criterion* (with respect to G) if $\tau(x-g_0, g_0-g) \geqslant 0$ for each $g \in G$, and the *strict Kolmogorov criterion* (with respect to G) if $\tau(x-g_0, g_0-g) > 0$ for each $g \in G\setminus\{g_0\}$. In the literature these criteria are given in another (equivalent) form. Namely (see, e.g., [15]), the pair (g_0, x) is said to satisfy the Kolmogorov criterion (similarly for the other criterion) if for each $g \in G$, $\min\{f(g-g_0) \mid f \in \text{ex}\, A(x-g_0)\} \leqslant 0$, where $\text{ex}\, A(x-g_0)$ is the set of the extreme points of $A(x-g_0)$. One can see immediately that they are equivalent using formula (1.9) below (see the first equality in [13, 18] and the second in [2]). For $x, y \in E$ we have

$$\tau(x, y) = \max\{f(y) \mid f \in A(x)\} = \max\{f(y) \mid f \in \text{ex}\, A(x)\}. \tag{1.9}$$

A set $G \subset E$ is called a *sun* (see, e.g., [20]) if for each $x \in E$ and $g_0 \in P_G(x)$, we have $g_0 \in P_G(\alpha x + (1-\alpha) g_0)$ for each $\alpha \geqslant 1$. Notice that for $0 \leqslant \alpha \leqslant 1$, the relation $g_0 \in P_G(\alpha x + (1-\alpha) g_0)$ holds for any set G. The Kolmogorov criterion was used to characterize the elements of best approximation when G is a sun, since Brosowski [3] proved the following result:

1.1. Theorem [3]. *A set $G \subset E$ is a sun if and only if for each $x \in E\setminus\bar{G}$, $g_0 \in G$, the following statements are equivalent:*

(i) $g_0 \in P_G(x)$,

(ii) (g_0, x) *satisfies the Kolmogorov criterion.*

Notice that (ii) ⇒ (i) is valid for an arbitrary set G.

We shall need the following simple, but useful, results.

1.2. Lemma [15]. *Let G be a subset of E, $x \in E\setminus\bar{G}$ and $g_0 \in G$. If (g_0, x) satisfies the strict Kolmogorov criterion, then $P_G(x) = \{g_0\}$.*

When G is a linear subspace of E, we denote by $P_G^{-1}(0) = \{x \in E \mid 0 \in P_G(x)\}$.

1.3. Proposition [5]. *A linear subspace G of E is Chebyshev if and only if $G \oplus P_G^{-1}(0) = E$.*

2. Spaces with Property (C)

2.1. Definition [8]. The space E is called with *property* (C) if for each $x \in S_E$ and each $a \in A(x)_\perp$, $\|a\| = 1$, we have $a = (z_1 - z_2)/2$, for some $z_i \in S_E$ with $A(x) \subset A(z_i)$, $i = 1, 2$. Notice that by (1.2), $z_i \in N(x)$ and we have $z_i = x + a_i$ for some $a_i \in A(x)_\perp$, $i = 1, 2$.

Geometrically, the space E has property (C), if for each $x \in S_E$ the closed linear subspace $N(x)$ of E has the property described as follows. Since x is a smooth point of $S_{N(x)}$, there exists a unique $\varphi \in S_{N(x)^*}$ such that $\varphi(x) = \|x\| = 1$. Let $F_x = \{z \in S_{N(x)} \mid \varphi(z) = 1\}$. Then F_x is a face of $S_{N(x)}$ with $x \in F_x$. Property (C) requires that each $a \in A(x)_\perp$ $(= \{z \in N(x) \mid \varphi(z) = 0\})$, $\|a\| = 1$, to be in the middle of a segment with an end-point in F_x and the other in $-F_x$.

2.2. *Remark.* If E has property (C), then for each $x \in S_E$ and each $a \in A(x)_\perp$, $\|a\| \leqslant 1$, we have $a = (z_1 - z_2)/2$ for some $z_i \in S_E$ with $A(x) \subset A(z_i)$, $i = 1, 2$.

In [8] we have shown that the spaces $C(Q)$ and $L^1(T, \mu)$, $(L^1)^* \equiv L^\infty$, have property (C). Using [10, Lemma 2], the assumption $(L^1)^* \equiv L^\infty$ can be deleted. (We recall that for $x \in S_E$ and $a \in A(x)_\perp$, $\|a\| = 1$, we can choose in Definition 2.1, for $E = C(Q)$, $z_1 = x(1 - |a|) + a$, $z_2 = x(1 - |a|) - a$ and for $E = L^1(T, \mu)$, $z_1 = a + |a| \operatorname{sign} x$, $z_2 = -a + |a| \operatorname{sign} x$.)

If A is a closed set in Q, then the following subspace of $C(Q)$, $I_A = \{x \in C(Q) \mid x|A = 0\}$ has property (C), the proof being similar with that for $C(Q)$. Hence the space $C_0(T)$ of all real-valued functions on the locally compact space T, vanishing at infinity, endowed with the sup norm, has property (C). In particular, c_0 has this property. Since property (C) is invariant under linearly isometries, it follows that $L^\infty(T, \mu)$ has property (C).

2.3. *Remark.* No smooth or strictly convex space E, $\dim E \geqslant 2$, has property (C).

2.4. *Remark.* Property (C) behaves badly with respect to the heredity. We shall show below (see Remark 2.12) that the space $C^1([a, b], \nu)$, ν the Lebesgue measure, has not property (C), though it is a dense subspace of $L^1([a, b], \nu)$ which has this property. If E is an arbitrary 2-dimensional space, then E has property (C) if and only if its unit ball is a parallelogram. Hence by [10, Proposition 1, Theorem 7] and Remark 2.3 above, it follows that when E is a space with property (C), $\dim E \geqslant 3$, then there exists a 2-dimensional subspace of E which has not property (C).

The main result of this section is the following characterization of a semi-Chebyshev set G, when G is a sun in a space with property (C).

2.5. Theorem. *Let E be a space with property (C) and G a sun of E. Then G is semi-Chebyshev if and only if for each $x \in E \backslash \bar{G}$ and each $g_0 \in P_G(x)$ we have $(G - g_0) \cap A(x - g_0)_\perp = \{0\}$.*

Proof. Suppose there are $x \in E \backslash \bar{G}$, and $g_0 \in P_G(x)$ such that for some $g_1 \in G$, $g_1 \neq g_0$ we have $g_1 - g_0 \in (G - g_0) \cap A(x - g_0)_\perp$. Since G is a sun, we can suppose $\|x-g_0\|=1$. The space E having property (C), there exist $z_i \in S_E$ with $A(x-g_0)\subset A(z_i)$, $i=1,2$, such that

$$\frac{g_1-g_0}{\|g_1-g_0\|}=\frac{z_1-z_2}{2}. \tag{2.1}$$

Since $g_0 \in P_G(x)$, by Theorem 1.1, we have $\tau(x-g_0, g_0-g)\geqslant 0$ for each $g\in G$. Then by $A(x-g_0)\subset A(z_i)$, $i=1,2$, and (1.9), it follows $\tau(z_i, g_0-g)\geqslant 0$ for each $g\in G$, whence $0\in P_{G-g_0}(z_i)$, $i=1,2$, and since $G-g_0$ is a sun, $0\in P_{G-g_0}((\|g_1-g_0\|/2)z_i)$, $i=1,2$. Hence, by $z_i\in S_E$, $i=1,2$, and (2.1), we have

$$\operatorname{dist}\left(\frac{\|g_1-g_0\|}{2}z_1, G-g_0\right)=\left\|\frac{\|g_1-g_0\|}{2}z_2\right\|=\left\|\frac{\|g_1-g_0\|}{2}z_1-(g_1-g_0)\right\|.$$

Therefore $0, g_1-g_0\in P_{G-g_0}((\|g_1-g_0\|/2)z_1)$, that is, $G-g_0$ is not semi-Chebyshev, and so G is not semi-Chebyshev. This proves the "only if" portion of the theorem.

To prove the "if" portion, suppose there is $x\in E\setminus\bar{G}$ such that $g_0, g_1\in P_G(x)$, $g_0\neq g_1$. By Theorem 1.1 it follows that $\tau(x-g_0, g_0-g)\geqslant 0$ for each $g\in G$. For $g=g_1$ we must have $\tau(x-g_0, g_0-g_1)=0$ since otherwise the pair $(0, x-g_0)$ satisfies the strict Kolmogorov criterion with respect to the set $L_1=\{\alpha(g_1-g_0)\mid \alpha\geqslant 0\}$, whence by Lemma 1.2, $P_{L_1}(x-g_0)=\{0\}$; on the other hand, by $g_0, g_1\in P_G(x)$, we have

$$\|x-g_0\|=\|x-g_1\|=\|(x-g_0)-(g_1-g_0)\| \tag{2.2}$$

hence $0\neq g_1-g_0\in P_{L_1}(x-g_0)$, contradicting $P_{L_1}(x-g_0)=\{0\}$. Therefore $\tau(x-g_0, g_0-g_1)=0$, whence by (1.1) we get $\tau(x-g_0, g_1-g_0)\geqslant 0$. Then the pair $(0, x-g_0)$ satisfies the Kolmogorov criterion with respect to the 1-dimensional subspace $[g_1-g_0]$ and so $0\in P_{[g_1-g_0]}(x-g_0)$. By (2.2) we also have $g_1-g_0\in P_{[g_1-g_0]}(x-g_0)$. Let $y=2(x-g_0)-(g_1-g_0)$. Then $P_{[g_1-g_0]}(y)=2P_{[g_1-g_0]}(x-g_0)-(g_1-g_0)$, and since $0, g_1-g_0\in P_{[g_1-g_0]}(x-g_0)$, it follows $\pm(g_1-g_0)\in P_{[g_1-g_0]}(y)$ whence, since this latter set is convex, $0\in P_{[g_1-g_0]}(y)$. Therefore we have

$$\|y\|=\|y-(g_1-g_0)\|=\|y+(g_1-g_0)\|. \tag{2.3}$$

By $0\in P_{[g_1-g_0]}(y)$ and Theorem 1.1 we get $\tau(y, \pm(g_1-g_0))\geqslant 0$. If $\tau(y, g_1-g_0)>0$, then the pair $(0, y)$ satisfies the strict Kolmogorov criterion with respect to the set $L_2=\{\alpha(g_1-g_0)\mid \alpha\leqslant 0\}$, whence by Lemma 1.2, $P_{L_2}(y)=\{0\}$. By (2.3) we get $g_0-g_1\in P_{L_2}(y)$, a contradiction. So, $\tau(y, g_1-g_0)=0$, and in a similar way, using the set L_1 and (2.3), it follows that $\tau(y, g_0-g_1)=0$. By (1.3) we obtain that $g_1-g_0\in A(y)$. Let

$w = y + g_0$. Then $w = 2x + (1-2)g_1$ and since G is a sun and $g_1 \in P_G(x)$, it follows $g_1 \in P_G(w)$. Using (2.3) we have $\|w - g_1\| = \|(y + g_0) - g_1\| = \|y\| = \|(y + g_0) - g_0\| = \|w - g_0\|$, and so $g_0 \in P_G(w)$. Since $w - g_0 = y$ we get $0 \neq g_1 - g_0 \in (G - g_0) \cap A(w - g_0)_{\perp}$, which completes the proof of the theorem.

We notice that in the proof of the "if" part of Theorem 2.5 we have not used the hypothesis that E has property (C), so we have a sufficient condition for a sun G in an arbitrary space to be semi-Chebyshev.

An immediate consequence of Theorem 2.5 is

2.6. COROLLARY. *Let E be a space with property (C) and G a linear subspace of E. Then G is semi-Chebyshev if and only if $A(x)_{\perp} \cap G = \{0\}$, for each $x \in P_G^{-1}(0)$.*

The necessity condition in Corollary 2.6 can be improved. Indeed, an easy consequence of Corollary 2.6 gives

2.7. COROLLARY [8]. *Let E be a space with property (C) and G a linear subspace of E. Then G is semi-Chebyshev if and only if $N(x) \cap G = \{0\}$ for each $x \in P_G^{-1}(0)$.*

Even when G is a convex set, we can not improve the "only if" condition in Theorem 2.5 by the condition $N(x - g_0) \cap (G - g_0) = \{0\}$, $x \in E \setminus \bar{G}$, $g_0 \in P_G(x)$, as simple examples show.

For $E = C(Q)$ (resp. $E = L^1(T, \mu)$) by Corollary 2.6 and (1.4) (resp. Corollary 2.7 and (1.5)) we obtain the following result of Cheney and Wulbert [5, Theorem 10] (resp. [5, Theorem 21]), which states that a linear subspace $G \subset C(Q)$ (resp. $G \subset L^1(T, \mu)$) is semi-Chebyshev if and only if 0 is the only element in G which vanishes on an α_G-set (resp. a β_G-set), where an α_G-set is any set of the form crit x for some $x \in P_G^{-1}(0)$ (resp. a β_G-set is any set of the form $Z(x)$ for some $x \in P_G^{-1}(0)$).

Using (1.8) and Corollary 2.6 we obtain a result for l^∞ of the same form as those of Cheney and Wulbert mentioned above.

2.8. COROLLARY. *A linear subspace $G \subset l^\infty$ is semi-Chebyshev if and only if 0 is the only element in G whose coordinates vanish on I_x and tend to zero for sequences $(n_k) \in \mathscr{N}_x$, for $x \in P_G^{-1}(0)$.*

Using Theorem 2.5 we can also obtain results of a similar form with those above, when G is a sun in $C(Q)$, $L^1(T, \mu)$, l^∞.

By Corollary 2.8 we can easily prove the following result of Phelps ([17, p. 251]; actually Phelps considered the space $L^\infty(T, \mu)$): Let $g = (\gamma_n) \in l^\infty$, $\|g\| = 1$. Then the 1-dimensional subspace $G = [g]$ of l^∞ is Chebyshev if and only if $\inf|\gamma_n| > 0$. Indeed, if $\inf|\gamma_n| > 0$, then by Corollary 2.8, G is

Chebyshev. Suppose now $\inf|\gamma_n|=0$. If there is an index n_0 such that $\gamma_{n_0}=0$, then for $x=(\xi_n)$, $\xi_{n_0}=1$ and $\xi_n=0$ for $n\neq n_0$, we have $x\in P_G^{-1}(0)$, $I_x=\{n_0\}$ and $\mathscr{I}_x=\varnothing$. Since the element $g\in G$ vanishes on I_x, by Corollary 2.8, G is not Chebyshev. If $\gamma_n\neq 0$ for all n, then there exists a subsequence (n_k) such that $\lim|\gamma_{n_k}|=0$, and we can suppose $\gamma_{n_k}>0$ for all k. Let $x=(\xi_n)\in l^\infty$ be defined by $\xi_{n_k}=1-\gamma_{n_k}$, $k=1,2,\dots$, and $\xi_i=0$ for $i\notin(n_k)$. Then $\|x\|=1$, $x\in P_G^{-1}(0)$, and we have $I_x=\varnothing$ and $\mathscr{I}_x$ is the set of all subsequences of (n_k). Again by Corollary 2.8, G is not Chebyshev.

Theorem 2.5 can be used to obtain necessary conditions for a sun G to be Chebyshev in some particular infinite-dimensional spaces with property (C). The condition $0\in G$ in the following is not a restrictive one, since G is a Chebyshev sun if and only if $G-g$ is a Chebyshev sun, $g\in G$.

2.9. COROLLARY. *Let E be an infinite-dimensional space with property (C) such that for each $x\in S_E$,* $\operatorname{codim} N(x)<\infty$, *and let G be a sun in E, $0\in G$. If $M=\operatorname{sp}\{G\}\neq E$ is infinite-dimensional and G contains an interior point relative to M, then G is not Chebyshev. In particular, such a space E has no infinite-dimensional Chebyshev subspaces $\neq E$.*

Proof. Suppose all the conditions in Corollary 2.9 are satisfied and G is Chebyshev. Let $g_0\in G$ be such that $\{m\in M \mid \|m-g_0\|<\varepsilon\}\subset G$ for some $\varepsilon>0$. In [9] it was shown that for any proximinal set G in an arbitrary space E the set $\mathscr{A}_G=\{g\in \operatorname{bd} G \mid \text{there exists } x\in E\backslash G \text{ with } g\in P_G(x)\}$ is dense in $\operatorname{bd} G$. Since $M\neq E$, we have $g_0\in \operatorname{bd} G$ and so there exists $g_1\in \operatorname{bd} G$, $\|g_1-g_0\|<\varepsilon$ such that $g_1\in P_G(x)$ for some $x\in E\backslash G$. Then there is $\delta>0$ such that $\{m\in M \mid \|m-g_1\|<\delta\}\subset G$, whence $\{m\in M \mid \|m\|<\delta\}\subset (G-g_1)$. Since $\dim M=\infty$ and $\operatorname{codim} A(x-g_1)_{\perp}<\infty$, we have $A(x-g_1)_{\perp}\cap(G-g_1)\neq\{0\}$, which contradicts Theorem 2.5.

The space $E=c_0$ satisfies the conditions of Corollary 2.9. So by this result we obtain the known fact (see, e.g., [19]) that c_0 has no infinite-dimensional Chebyshev subspaces $\neq c_0$.

With a proof similar to that of Corollary 2.9, one can show

2.10. COROLLARY. *Let E be an infinite-dimensional space with property (C), such that for each $x\in S_E$,* $\dim N(x)=\infty$, *and let G be a sun in E, $0\in G$. If $M=\operatorname{sp}\{G\}\neq E$ is closed and of finite codimension, and G contains an interior point relative to M, then G is not Chebyshev. In particular, such a space E has no Chebyshev subspaces of finite codimension $\neq E$.*

The space $E=L^1(T,\mu)$, where (T,μ) has no atoms, satisfies the conditions of Corollary 2.10, whence we obtain to known fact (see, e.g., [19]) that if (T,μ) has no atoms then $L^1(T,\mu)$ has no Chebyshev subspaces of finite codimension $\neq L^1(T,\mu)$.

By Proposition 1.3 and Corollary 2.7 we get the following necessary condition on a space with property (C) in order to have a Chebyshev subspace of codimension n.

2.11. COROLLARY. *Let E be a space with property (C) which has a Chebyshev subspace of codimension n. Then there exist at least n linearly independent elements $x_1, \ldots, x_n \in S_E$, such that* $\dim N(x_i) \leqslant n$, $i = 1, \ldots, n$.

Hence for $E = L^1(T, \mu)$ we obtain the "only if" part of the following result (see, e.g., [19]): The space $L^1(T, \mu)$ has a Chebyshev subspace of codimension n if and only if (T, μ) has at least n atoms. For $E = C([a, b])$ we obtain the known fact (see, e.g., [19]) that it has no Chebyshev subspaces of codimension n for $2 \leqslant n < \infty$ (since in this space for $x \equiv 1$, by (1.4), $N(x) = [x]$ and for each $y \in S_E$, $y \neq \pm x$, we have $\dim N(y) = \infty$).

Finally, we give the following consequence of Corollary 2.7.

2.12. *Remark.* The space $C^1([a, b], \nu)$, ν the Lebesgue measure has not property (C). Indeed, suppose it has this property, and let G be the set of all algebraic polynomials of degree $\leqslant n$ defined on $[a, b]$. By Jackson's Theorem (see, e.g., [19]), G is a Chebyshev subspace of $C^1([a, b], \nu)$. Let p_{n+1} be a polynomial of degree $n+1$ defined on $[a, b]$, and let $p_n \in G$ such that $P_G(p_{n+1}) = \{p_n\}$. Then $x = p_{n+1} - p_n \in P_G^{-1}(0)$ is a polynomial of degree $n+1$: it has at most $n+1$ zeros in $[a, b]$, whence by (1.5) for C^1, we get $N(x) = C^1([a, b], \nu)$. Then $N(x) \cap G = G$, in contradiction with Corollary 2.7. Therefore $C^1([a, b], \nu)$ has not property (C).

3. SPACES WITH PROPERTY (A)

We remarked in [10] that for each $x, y \in E$ we have

$$\tau(x, y) + \tau(x, -y) \leqslant 2 \operatorname{dist}(y, N(x)). \tag{3.1}$$

3.1. DEFINITION [10]. A space E is called with *property* (A) if $\tau(x, y) + \tau(x, -y) = 2 \operatorname{dist}(y, N(x))$ for each $x, y \in S_E$ (equivalently, for each $x, y \in E$).

In [10] we asked whether the space $C(Q)$ has property (A). An affirmative answer is given in the next result.

3.2. THEOREM. *The space $E = C(Q)$ has property (A). Moreover, for each $x \in S_E$, $N(x)$ is proximinal.*

Proof. Let $x, y \in S_E$ such that $y \notin N(x)$. By (1.9) we have

$$\tau(x, y) = \max\{y(q) \operatorname{sign} x(q) \mid q \in \operatorname{crit} x\}. \tag{3.2}$$

Let us put

$$\lambda = \frac{\tau(x,y) - \tau(x,-y)}{2} \tag{3.3}$$

and denote by $I_1 = \{q \in Q \mid x(q) = 1\}$ and $I_2 = \{q \in Q \mid x(q) = -1\}$. Define the following continuous function on crit x:

$$\begin{aligned} v(q) &= \lambda - y(q), \qquad q \in I_1, \\ &= -\lambda - y(q), \qquad q \in I_2. \end{aligned} \tag{3.4}$$

Using (3.2) and (3.3) one can easily show that $|v(q)| \leqslant (\tau(x,y) + \tau(x,-y))/2$ for each $q \in$ crit x. By Tietze's Theorem there exists a continuous function w on Q, such that $w(q) = v(q)$ for each $q \in$ crit x and $\|w\| \leqslant (\tau(x,y) + \tau(x,-y))/2$. Let $z = w + y$. Since for $q \in$ crit x, $w(q) = v(q)$, by (3.4) and (1.4) it follows $z \in N(x)$. We have $\text{dist}(y, N(x)) \leqslant \|y - z\| = \|w\| \leqslant (\tau(x,y) + \tau(x,-y))/2$, whence by (4.1), $\text{dist}(y, N(x)) = (\tau(x,y) + \tau(x,-y))/2 = \|y - z\|$, which show that $C(Q)$ has property (A) and that $N(x)$ is proximinal.

With a proof similar to that above, it follows that for any closed set $A \subset Q$, I_A has property (A), and for each $x \in I_A$, $N(x)$ is proximinal. In particular, $C_0(T)$ has these properties. Since property (A) is invariant under linear isometries, the space $L^1(T,\mu)$ has property (A) and $N(x)$ is proximinal for each $x \in L^1(T,\mu)$. In [10] we remarked that any smooth space has property (A), and we proved that the spaces $L^1(T,\mu)$ and $C^1(Q,\nu)$ have property (A). We noticed there that for each $x \in L^1(T,\mu)$, $N(x)$ is proximinal. Property (A) (as well as property (C)) behaves badly with respect to the heredity [10].

We do not know an example of a space with property (C) but without (A). Notice that property (C) implies that for each $x \in S_E$, any face of S_E containing x has the diameter 0 or 2, while property (A) implies that for each $x \in S_E$, the diameter of the set $A(x) \subset E^*$ is 0 or 2.

3.3. Theorem. *Let E be a normed linear space. The following assertions are equivalent:*

(i) *E has property (A).*

(ii) *For each sun G of E, each $x \in E \setminus \bar{G}$ and $g_0 \in G$, the condition*

$$\begin{aligned} &\tau(x - g_0, g - g_0) + \tau(x - g_0, g_0 - g) \\ &\qquad \leqslant 2\,\text{dist}(g - g_0, N(x - g_0)) \qquad (g \in G), \end{aligned} \tag{3.5}$$

is a necessary and sufficient condition that $g_0 \in P_G(x)$.

(iii) *For each linear subspace* $G \subset E$, *each* $x \in E \setminus \bar{G}$ *and* $g_0 \in G$, *the condition*

$$|\tau(x - g_0, g - g_0) - \tau(x - g_0, g_0 - g)| \leqslant 2\,\mathrm{dist}(g - g_0, N(x - g_0)) \qquad (g \in G), \tag{3.6}$$

is a necessary and sufficient condition that $g_0 \in P_G(x)$.

Proof. (i) $\Rightarrow$ (ii). Let G be a sun of E, $x \in E \setminus \bar{G}$ and $g_0 \in G$. If $g_0 \in P_G(x)$, by Theorem 1.1 we have $\tau(x - g_0, g_0 - g) \geqslant 0$ for each $g \in G$. Hence, using also (3.1), we have $\tau(x - g_0, g - g_0) - \tau(x - g_0, g_0 - g) \leqslant \tau(x - g_0, g - g_0) + \tau(x - g_0, g_0 - g) \leqslant 2\,\mathrm{dist}(g - g_0, N(x - g_0))$ for each $g \in G$, so we have (3.5). Note that for the necessity part we have not used the hypothesis (i). If E has property (A) and (3.5) holds, then clearly the pair (g_0, x) satisfies the Kolmogorov criterion, whence by Theorem 1.1, $g_0 \in P_G(x)$.

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) follow by [10, Theorem 3].

In the class of spaces with property (A), statement (ii) or (iii) does not say more than Theorem 1.1, so only the implication (iii) $\Rightarrow$ (i) is worth noting.

When $E = L^1(T, \mu)$, if we replace in (3.6) resp. (3.5) the expressions given by (1.6) and (1.7), then (iii) is a result of Kripke and Rivlin [12, Theorem 1.3] and (ii) is a result of Deutsch [6].

3.4. *Remark.* It was observed in [15] that when G is a set in an arbitrary normed linear space E such that for each $x \in E \setminus \bar{G}$ and $g_0 \in P_G(x)$ the pair (g_0, x) satisfies the strict Kolmogorov criterion, then G is a semi-Chebyshev sun. This follows by Theorem 1.1 and Lemma 1.2.

Using Remark 3.4 and Theorem 3.3 one can easily prove

3.5. Proposition. *Let E be a space with property (A) and G a set in E. The following assertions are equivalent*:

(i) *For each* $x \in E \setminus \bar{G}$ *and* $g_0 \in G$, *the pair* (g_0, x) *satisfies the strict Kolmogorov criterion.*

(ii) *For each* $x \in E \setminus \bar{G}$, $g_0 \in P_G(x)$, $g \in G \setminus \{g_0\}$, *we have*

$$\tau(x - g_0, g - g_0) - \tau(x - g_0, g_0 - g) < 2\,\mathrm{dist}(g - g_0, N(x - g_0)). \tag{3.7}$$

For $E = L^1(T, \mu)$, if we replace in (3.7) the expressions given by (1.6) and (1.7), then Proposition 3.5 gives [15, Theorem 2.8, (2) $\Leftrightarrow$ (4)].

By Remark 3.4, the condition (i) in Proposition 3.5 implies that G is a semi-Chebyshev sun in an arbitrary normed linear space. The converse is not always true even when E has property (A) and G is a linear subspace, as simple examples in a strictly convex and smooth space show. We shall see in the next section that under some additional assumptions on E, this converse statement is true.

4. Spaces with Both Properties (C) and (A)

As one can see by Sections 2 and 3, the following spaces have both properties (C) and (A): $C(Q)$, I_A (A a closed subset of Q), $C_0(T)$, $L^{\prime}(T,\mu)$, $L^1(T,\mu)$. By Theorem 3.1 and the comments after the proof of this theorem, in each of the above concrete spaces, $N(x)$ is proximinal for each $x \in E$.

4.1. Theorem. *Let E be a normed linear space with properties (C) and (A) such that for each $x \in E$, $N(x)$ is proximinal, and let G be a subset of E. The following assertions are equivalent*:

(i) *G is a semi-Chebyshev sun.*

(ii) *For each $x \in E \backslash \bar{G}$ and $g_0 \in P_G(x)$, the pair (g_0, x) satisfies the strict Kolmogorov criterion.*

Proof. By Remark 3.3 we must only show that (i) $\Rightarrow$ (ii). Let G be a semi-Chebyshev sun and suppose there are $x \in E \backslash \bar{G}$ and $g_0 \in P_G(x)$ such that $\tau(x-g_0, g_0-g_1) \leqslant 0$ for some $g_1 \in G \backslash \{g_0\}$. Without loss of generality we can suppose $g_0 = 0$ and $\|x\| = 1$. Since $0 \in P_G(x)$, by Theorem 1.1, it follows $\tau(x, -g_1) = 0$. By (1.1) we have $\tau(x, g_1) \geqslant 0$. Then $\tau(x, g_1) > 0$, since otherwise by (1.3), $0 \neq g_1 \in A(x) \cap G$, whence by Theorem 2.5 G is not semi-Chebyshev, a contradiction. Hence by (1.9) there exist $f_1, f_2 \in A(x)$ such that

$$\tau(x, g_1) = f_1(g_1) > 0, \tag{4.1}$$

$$\tau(x, -g_1) = f_2(-g_1) = 0. \tag{4.2}$$

Since $N(x)$ is proximinal, there is $y \in P_{N(x)}(g_1)$. By (1.2), $y = \lambda x + a$ for some $\lambda \in R$ and $a \in A(x)$. Hence, using the hypothesis on E to have property (A), and (4.1), (4.2), we get

$$\| g_1 - \lambda x - a \| = \operatorname{dist}(g_1, N(x)) = \frac{\tau(x, g_1)}{2} = \frac{f_1(g_1)}{2} > 0. \tag{4.3}$$

Since $f_1 \in A(x)$, we have by (4.3) that $f_1(g_1) - \lambda = f_1(g_1 - \lambda x - a) \leqslant \| g_1 - \lambda x - a \| = f_1(g_1)/2$, and so $\lambda \geqslant f_1(g_1)/2 = \| g_1 - \lambda x - a \|$. By (4.2) and $f_2 \in A(x)$ we have that $\lambda = f_2(\lambda x + a - g_1) \leqslant \| \lambda x + a - g_1 \|$, and so $\lambda = \| \lambda x + a - g_1 \| > 0$.

If $a = 0$, then $\lambda = \| \lambda x - g_1 \| = \| \lambda x \|$, and since $0 \in P_G(x)$ and G is a sun, we get $0, g_1 \in P_G(\lambda x)$, which contradicts (i). Therefore $a \neq 0$, and using the hypothesis that E has property (C), we have $a/\|a\| = (z_1 - z_2)/2$, where $z_i \in S_E$ and $A(x) \subset A(z_i)$, $i = 1, 2$. Now, $z_i = x + a_i$ for some $a_i \in A(x)$,

$i = 1, 2$, and so $a/\|a\| = (a_1 - a_2)/2$. Since $0 \in P_{A(x)_\perp}(x)$ and $\|x + a_i\| = 1 = \|x\|$, $i = 1, 2$, it follows that

$$\|x + \alpha a_i\| = 1 \qquad \text{for each } 0 \leqslant \alpha \leqslant 1, i = 1, 2. \tag{4.4}$$

Let

$$z = (\lambda + \|a\|)x + \frac{\|a\|}{2} a_1.$$

Then by (4.4) we have

$$\|z\| = (\lambda + \|a\|) \left\| x + \frac{\|a\|}{2(\lambda + \|a\|)} a_1 \right\| = \lambda + \|a\|.$$

Now, for each $f \in A(x)$ we have $f(z) = \lambda + \|a\| = \|z\|$, and so $A(x) \subset A(z)$. Since $0 \in P_G(x)$, by Theorem 1.1 we have $\tau(x, -g) \geqslant 0$ for each $g \in G$, whence by (1.9) and $A(x) \subset A(z)$ we have also $\tau(z, -g) \geqslant 0$ for each $g \in G$, hence $0 \in P_G(z)$. On the other hand,

$$\begin{aligned}\|z - g_1\| &\leqslant \|\lambda x + a - g_1\| + \left\| \|a\| x + \frac{\|a\|}{2} a_1 - a \right\| \\ &= \lambda + \left\| \|a\| x + \frac{\|a\|}{2} a_1 - \frac{\|a\|}{2} (a_1 - a_2) \right\| \\ &= \lambda + \|a\| \left\| x + \frac{a_2}{2} \right\| = \lambda + \|a\| = \|z\|.\end{aligned}$$

Therefore, $0, g_1 \in P_G(z)$ which contradicts (i) and completes the proof.

For $E = L^1(T, \mu)$, the equivalence (i) $\Leftrightarrow$ (ii) in Theorem 4.1 was proved by Nürnberger [15, Theorem 2.8, (1) $\Leftrightarrow$ (2)]. Replacing condition (i) by "G is semi-Chebyshev," the equivalence of this condition with (ii) was proved in [15, Theorem 2.4, (1) $\Leftrightarrow$ (3)] for $E = C(Q)$ and G a finite dimensional convex set. Theorem 4.1 generalizes this result for arbitrary suns of $C(Q)$. The equivalence (i) $\Leftrightarrow$ (ii) in Theorem 4.1 is also true for l_1, $C_0(T)$, $L^1(T, \mu)$.

By [15, Remark 3.3] and Theorem 4.1, we obtain immediately the following result.

4.2. COROLLARY. *Let E be a space with properties (C) and (A), such that $N(x)$ is proximinal for each $x \in E$, and let G be a set in E.*

(i) *If G is a finite-dimensional Chebyshev subspace of E, then G is a strongly Chebyshev subspace.*

(ii) *If G is a one-dimensional convex Chebyshev set of E, then G is strongly Chebyshev.*

(iii) *If G is a finite-dimensional semi-Chebyshev convex cone with vertex in the origin of E, then for each $x \in E$ with $0 \in P_G(x)$ the element 0 is a strongly unique element of best approximation.*

Statement (i) in Corollary 4.2 has been proved by Newman and Shapiro [14] for $E = C(Q)$, by Ault *et al.* [1] for $E = C_0(T)$, by Wulbert [21] for $E = L^1(T, \mu)$ and by Nürnberger [15] for $E = l_1$. Statements (ii) and (iii) in Corollary 4.2 have been proved by Nürnberger for $E = l_1$ or $L^1(T, \mu)$.

Corollary 4.2 can be used to obtain sufficient conditions for the metric projection to be pointwise Lipschitzian since Cheney [4, p. 82] showed that pointwise Lipschitzian continuity follows from strong unicity properties. For $E = l_1$ or $L^1(T, \mu)$, results on pointwise Lipschitzian metric projection have been given in [15].

It is our belief that some other results on best approximation in concrete spaces could be formulated and proved in the framework of spaces with property (C) or (and) property (A).

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